

A Direct Connection Between the Bergman and Szegő Kernels¹²

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Abstract: *We use Stokes's theorem to establish an explicit and concrete connection between the Bergman and Szegő projections on the disc, the ball, and on strongly pseudoconvex domains.*

1 Introduction

Two of the most classical and well established reproducing formulas in complex analysis are those of S. Bergman and G. Szegő. The first of these is a formula for the Bergman space, and the associated integral lives on the interior of the domain in question. The latter of these is a formula for the Hardy space, and the associated integral lives on the boundary of the domain. For formal reasons, the Bergman integral gives rise to a projection from $L^2(\Omega)$ to $A^2(\Omega)$ (the Bergman space); likewise, the Szegő integral gives rise to a projection from $L^2(\partial\Omega)$ to $H^2(\Omega)$ (the Hardy space).

Since both of the artifacts in question here are canonical, it is natural to suspect that there is some relationship between the two integral formulas. After all, they both reproduce functions that are continuous on the closure of the domain and holomorphic on the interior. In the present paper we establish such a connection—very explicitly—on a variety of domains in \mathbb{C}^1 and \mathbb{C}^n . This is done by way of a moderately subtle calculation using Stokes's theorem. The calculation itself has some intrinsic interest, but the main point is the equality of the canonical integrals and the associated projections.

¹**Key Words:** harmonic analysis, several complex variables, Bergman kernel, Szegő kernel, reproducing kernels

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2 The Case of the Disc

Let D be the unit disc in \mathbb{C} . In this context, the Szegő kernel is

$$S(z, \zeta) = \frac{1}{2\pi} \cdot \frac{1}{1 - z \cdot \bar{\zeta}}$$

and the Bergman kernel is

$$K(z, \zeta) = \frac{1}{\pi} \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^2}.$$

Take f to be real analytic on a neighborhood of \bar{D} . Now we can calculate

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial D} f(\zeta) S(z, \zeta) d\sigma(\zeta) &= \frac{1}{2\pi} \int_{\partial D} f(\zeta) \cdot \frac{1}{1 - z \cdot \bar{\zeta}} \frac{[\bar{\zeta} d\zeta - \zeta d\bar{\zeta}]}{2i} \\ &= \frac{1}{4\pi i} \int_{\partial D} \frac{f(\zeta) \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\zeta - \frac{1}{4\pi i} \int_{\partial D} \frac{f(\zeta) \zeta}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \\ &\stackrel{(\text{Stokes})}{=} \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta + \frac{1}{4\pi i} \iint_D \frac{f(\zeta) \bar{\zeta} z}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta \\ &\quad - \frac{1}{4\pi i} \iint_D \frac{\partial(f \cdot \zeta)/\partial \zeta}{1 - z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} + \frac{1}{4\pi i} \iint_D \frac{\partial f/\partial \bar{\zeta} \cdot \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\ &= \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{f(\zeta) z \bar{\zeta}}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta \\ &\quad + \frac{1}{4\pi i} \iint_D \frac{f(\zeta) \bar{\zeta} z}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{\partial(f \cdot \zeta)/\partial \zeta}{1 - z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} \\ &\quad + \frac{1}{4\pi i} \iint_D \frac{\partial f/\partial \bar{\zeta} \cdot \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\ &= \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{\partial(f \cdot \zeta)/\partial \zeta}{1 - z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} \\ &\quad + \frac{1}{4\pi i} \iint_D \frac{\partial f/\partial \bar{\zeta} \cdot \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\ &= \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{(1 - z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{\partial f/\partial \bar{\zeta} \cdot \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\ &\quad - \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{1 - z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} + \frac{1}{4\pi i} \iint_D \frac{\partial f/\partial \bar{\zeta} \cdot \bar{\zeta}}{1 - z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{(1-z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{\partial f / \partial \zeta \cdot \zeta}{1-z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} \\
&\quad - \frac{1}{4\pi i} \iint_D \frac{f(\zeta)}{(1-z \cdot \bar{\zeta})^2} d\zeta \wedge d\bar{\zeta} + \frac{1}{4\pi i} \iint_D \frac{f(\zeta)z\bar{\zeta}}{(1-z \cdot \bar{\zeta})^2} d\zeta \wedge d\bar{\zeta} \\
&\quad + \frac{1}{4\pi i} \iint_D \frac{\partial f / \partial \bar{\zeta} \cdot \bar{\zeta}}{1-z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\
&= \frac{1}{2\pi i} \iint_D \frac{f(\zeta)}{(1-z \cdot \bar{\zeta})^2} d\bar{\zeta} \wedge d\zeta - \frac{1}{4\pi i} \iint_D \frac{\partial f / \partial \zeta \cdot \zeta}{1-z \cdot \bar{\zeta}} d\zeta \wedge d\bar{\zeta} \\
&\quad + \frac{1}{4\pi i} \iint_D \frac{f(\zeta)z\bar{\zeta}}{(1-z \cdot \bar{\zeta})^2} d\zeta \wedge d\bar{\zeta} + \frac{1}{4\pi i} \iint_D \frac{\partial f / \partial \bar{\zeta} \cdot \bar{\zeta}}{1-z \cdot \bar{\zeta}} d\bar{\zeta} \wedge d\zeta \\
&= A - B + C + D.
\end{aligned}$$

Certainly $A = \int_D f(\zeta) K(z, \zeta) dA(\zeta)$, where K is the Bergman kernel of the disc. So this is the Bergman projection. Now we claim that $-B + C + D \equiv 0$. If we can establish that assertion, then we will have seen directly, by way of Stokes's theorem, that the Szegő projection equals the Bergman projection (at least for functions real analytic on the closure).

First assume that f is holomorphic. We establish the claim by verifying it for $f(\zeta) = \zeta^k$, each $k = 0, 1, 2, \dots$. Indeed, in this case (expanding the kernel in a Neumann series and discarding terms that obviously integrate to zero by parity)

$$B = \frac{1}{4\pi i} \iint_D k z^k |\zeta|^{2k} d\bar{\zeta} \wedge d\zeta.$$

And a similar calculation shows that

$$C = \frac{1}{4\pi i} \iint_D k z^k |\zeta|^{2k} d\bar{\zeta} \wedge d\zeta.$$

And $D = 0$ because $\partial f / \partial \bar{\zeta} \equiv 0$. Thus $-B + C + D = 0$ as desired.

For any monomial containing some positive power of $\bar{\zeta}$, it is easy to see by parity (again using the Neumann series for the kernel) that the integrals B, C, D are equal to 0. Summing, we see that we have proved our result for any function f that is real analytic on a neighborhood of \bar{D} . But standard measure theory, together with the Weierstrass approximation theorem, enable us to pass from these functions to, for example, functions that are continuous on \bar{D} .

Thus we see by our calculation that the full Szegő projection is equal to the full Bergman projection on the disc D .

We treat the case of the Bergman and Szegő projections on the ball below.

Given Fefferman's asymptotic expansion for the Bergman kernel [FEF], and Boutet de Monvel/Sjöstrand's asymptotic expansion for the Szegő kernel [BOS],

one would expect a like calculation (up to a controllable error term) on a smoothly bounded, strongly pseudoconvex domain. Unfortunately we do not know enough about the canonical kernels on domains of finite type to be able to predict what will happen there. We explore the strongly pseudoconvex case below.

In a more recent work, Chen and Fu [CHF] have explored some new comparisons of the Bergman and Szegő kernels. A sample theorem is this:

Theorem 2.1 *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2 boundary. Then*

- (1) *For any $0 < a < 1$, there exists a constant $C > 0$ such that*

$$\frac{S(z, z)}{K(z, z)} \leq C \delta(z) |\log \delta(z)|^{n/a}.$$

- (2) *If there is a neighborhood U of $\partial\Omega$, a bounded, continuous plurisubharmonic function φ on $U \cap \Omega$, and a defining function ρ of Ω satisfying $i\partial\bar{\partial}\varphi \geq i\rho^{-1}\partial\bar{\partial}\rho$ on $U \cap \Omega$ as currents, then there exists constants $0 < a < 1$ and $C > 0$ such that*

$$\frac{S(z, z)}{K(z, z)} \geq C \delta(z) |\log \delta(z)|^{-1/a}.$$

These authors further show that, for a C^2 -bounded *convex* domain the quotient S/K is comparable to δ without any logarithmic factor.

The techniques used in this work are weighted estimates for the $\bar{\partial}$ operator (in the spirit of Hörmander's work [HOR]) and also an innovative use of the Diederich-Fornæss index (see [DIF]). We can say no more about the work here.

We turn next to an examination of the situation on the unit ball B in \mathbb{C}^n .

3 The Unit Ball in \mathbb{C}^n

For simplicity we shall in fact restrict attention to complex dimension 2. In that situation, the area measure $d\sigma$ on the boundary is given by

$$d\sigma = \frac{1}{16} \left[\zeta_1 d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 - \zeta_2 d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 + \bar{\zeta}_1 d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 - \bar{\zeta}_2 d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \right].$$

As a result, we have

$$\begin{aligned} \int_{\partial B} f(\zeta) S(z, \zeta) d\sigma(\zeta) &= \frac{1}{2\pi^2} \cdot \frac{1}{16} \iiint_{\partial B} f(\zeta) \frac{1}{(1 - z \cdot \bar{\zeta})^2} \left[\zeta_1 d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 - \zeta_2 d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \right. \\ &\quad \left. + \bar{\zeta}_1 d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 - \bar{\zeta}_2 d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \right] \\ &= \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \zeta_1}{(1 - z \cdot \bar{\zeta})^2} \cdot \zeta_1 d\zeta_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{32\pi^2} \iiint_B \frac{f}{(1-z \cdot \bar{\zeta})^2} d\zeta_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \zeta_2}{(1-z \cdot \bar{\zeta})^2} \cdot \zeta_2 d\zeta_2 \wedge d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{f}{(1-z \cdot \bar{\zeta})^2} d\zeta_2 \wedge d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \\
& + \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \bar{\zeta}_1}{(1-z \cdot \bar{\zeta})^2} \cdot \bar{\zeta}_1 d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\
& + \frac{1}{32\pi^2} \iiint_B \frac{f}{(1-z \cdot \bar{\zeta})^2} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\
& + \frac{1}{32\pi^2} \iiint_B \frac{2f \cdot \bar{\zeta}_1 z_1}{(1-z \cdot \bar{\zeta})^3} d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \bar{\zeta}_2}{(1-z \cdot \bar{\zeta})^2} \cdot \bar{\zeta}_2 d\bar{\zeta}_2 \wedge d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{f}{(1-z \cdot \bar{\zeta})^2} d\bar{\zeta}_2 \wedge d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{2f \cdot \bar{\zeta}_2 z_2}{(1-z \cdot \bar{\zeta})^3} d\bar{\zeta}_2 \wedge d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2.
\end{aligned}$$

Now we may group together like terms to obtain

$$\begin{aligned}
= & -\frac{1}{8\pi^2} \iiint_B \frac{f(\zeta)}{(1-z \cdot \bar{\zeta})^3} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \\
& + \frac{3}{16\pi^2} \iiint_B \frac{f(\zeta) \cdot (z \cdot \bar{\zeta})}{(1-z \cdot \bar{\zeta})^3} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \\
& + \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \zeta_1}{(1-z \cdot \bar{\zeta})^2} \cdot \zeta_1 d\zeta_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \zeta_2}{(1-z \cdot \bar{\zeta})^2} \cdot \zeta_2 d\zeta_2 \wedge d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \bar{\zeta}_1}{(1 - z \cdot \bar{\zeta})^2} \cdot \bar{\zeta}_1 d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\
& - \frac{1}{32\pi^2} \iiint_B \frac{\partial f / \partial \bar{\zeta}_2}{(1 - z \cdot \bar{\zeta})^2} \cdot \bar{\zeta}_2 d\bar{\zeta}_2 \wedge d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \\
& = -A + B + C - D + E - F.
\end{aligned}$$

Now $-A$ is just the usual Bergman integral on the ball B in \mathbb{C}^2 . And we can argue, just as on the disc, that the other terms cancel out (or are zero outright, just by parity). We have verified that the Szegő projection integral equals the Bergman projection integral on the unit ball $B\mathbb{B}\mathbb{C}^n$.

4 Strongly Pseudoconvex Domains

We again, for simplicity, restrict attention to \mathbb{C}^2 . In the seminal paper [FEF], Fefferman shows that, near a strongly pseudoconvex boundary point, the Bergman kernel may be written (in suitable local coordinates) as

$$\frac{2}{\pi^2} \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^3} + \mathcal{E}(z, \zeta),$$

where \mathcal{E} is an error term of strictly lower order (in the sense of pseudodifferential operators) than the Bergman kernel.

In the important paper [BOS], Boutet de Monvel and Sjöstrand show that, near a strongly pseudoconvex boundary point, the Szegő kernel may be written (in suitable local coordinates) as

$$\frac{1}{2\pi^2} \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^2} + \mathcal{F}(z, \zeta),$$

where \mathcal{F} is an error term of strictly lower order (in the sense of pseudodifferential or Fourier integral operators) than the Szegő kernel.

We now take advantage of these two asymptotic expansions to say something about the relationship between the Bergman and Szegő projections on a smoothly bounded strongly pseudoconvex domain.

Now fix a smoothly bounded, strongly pseudoconvex domain Ω with defining function ρ (see [KRA1] for this notion). Let U be a tubular neighborhood of $\partial\Omega$ and let V be a relatively compact subdomain of U that is also a tubular neighborhood of $\partial\Omega$. Let φ_j be a partition of unity that is supported in U and sums to be identically 1 on V . We assume that each φ_j has support so small that both the Fefferman and Boutet de Monvel/Sjöstrand expansions are valid

on the support of φ_j . Then we write

$$\begin{aligned} \int_{\partial\Omega} f(\zeta) S(z, \zeta) d\sigma(\zeta) &= \iiint_{\partial\Omega} f(\zeta) S(z, \zeta) \omega(\zeta) \\ &= \sum_j \iiint_{\partial\Omega} \varphi_j(\zeta) f(\zeta) S(z, \zeta) \omega(\zeta), \end{aligned}$$

where ω is the differential form that is equivalent to area measure on the boundary. And now, using Boutet de Monvel/Sjöstrand, and using the notable lemma of Fefferman [FEF] that says that a strongly pseudoconvex boundary point is the ball up to fourth order, one can write each term of this last sum as

$$\begin{aligned} \frac{1}{2\pi} \iiint_{\partial B} \tilde{\varphi}_j(\zeta) f(\zeta) \cdot \frac{1}{(1 - z \cdot \bar{\zeta})^2} \Big[\zeta_1 d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \\ - \zeta_2 d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 + \bar{\zeta}_1 d\bar{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 - \bar{\zeta}_2 d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\zeta_2 \Big] + \mathcal{G}, \end{aligned}$$

where the error term \mathcal{G} arises from approximating $\partial\Omega$ by ∂B , from approximating the Szegő kernel S by the kernel for the ball, by applying a change of variable to φ_j , and also by approximating ω by the differential form that we used on the ball.

Now we may carry out the calculations using Stokes's theorem just as in the last section to finally arrive at the assertion that the last integral equals

$$\iiint_B \tilde{\varphi}_j(\zeta) \frac{f(\zeta)}{(1 - z \cdot \bar{\zeta})^3} dV + \mathcal{H}.$$

We cannot make the error term \mathcal{H} disappear this time, but it is smoothly bounded hence negligible. Finally, we can use the Fefferman asymptotic expansion to relate this last integral to the Bergman projection integral on the strongly pseudoconvex domain Ω .

In summary, we have used Stokes's theorem to relate the Szegő projection integral on a smoothly bounded, strongly pseudoconvex domain to the Bergman projection integral on that domain. In this context, we do not get a literal equality. Instead we get an equality up to a controllable error term.

5 Concluding Remarks

Certainly one of the fundamental problems of the function theory of several complex variables is to understand the canonical kernels in as much detail as possible. This paper is a contribution to that program. In future papers we hope to explore the finite type case in \mathbb{C}^n and other more general domains as well.

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